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Advanced Linear Algebra (MA 409)
Problem Sheet - 19

Eigenvalues and Eigenvectors

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1. Label the following statements as true or false.

   (a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

   (b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

   (c) There exists a square matrix with no eigenvectors.

   (d) Eigenvalues must be nonzero scalars.

   (e) Any two eigenvectors are linearly independent.

   (f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

   (g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.

   (h) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.

   (i) Similar matrices always have the same eigenvalues.

   (j) Similar matrices always have the same eigenvectors.

   (k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

   (a) $V = \mathbb{R}^3,\ T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$, and

   $$\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

   (b) $V = P_2(\mathbb{R}),\ T(a+bx+cx^2) =$

   $$(-4a+2b-2c) - (7a+3b+7c)x + (7a+b+5c)x^2,$$

   and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$

   (c) $V = P_3(\mathbb{R}),\ T(a+bx+cx^2+dx^3) =$

   $$-d + (-c+d)x + (a+b-2c)x^2 + (-b+c-2d)x^3,$$

   and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$

   (d) $V = M_{2\times 2}(\mathbb{R}),\ T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$, and

   $$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

3. For each of the following matrices $A \in M_{n\times n}(F)$,

   (i) Determine all the eigenvalues of A.

   (ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

   (iii) If possible, find a basis for F^n consisting of eigenvectors of A.

   (iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   (a) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$

   (b) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbb{C}$

4. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

   (a) $V = \mathbb{R}^3$ and $T(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$

   (b) $V = P_1(\mathbb{R})$ and $T(ax + b) = (-6a + 2b)x + (-6a + b)$

   (c) $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$

   (d) $V = P_3(\mathbb{R})$ and $T(f(x)) = xf'(x) + f''(x) - f(2)$

   (e) $V = M_{2\times2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

   (f) $V = M_{2\times2}(\mathbb{R})$ and $T(A) = A^t + 2\cdot tr(A)\cdot I_2$

5. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

6. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(T)$, as follows: Choose any ordered basis β for V, and define $\det(T) = \det([T]_\beta)$.

   (a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

   (b) Prove that T is invertible if and only if $\det(T) \neq 0$.

   (c) Prove that if T is invertible, then $\det(T^{-1}) = [\det(T)]^{-1}$.

   (d) Prove that if U is also a linear operator on V, then $\det(TU) = \det(T)\cdot\det(U)$.

   (e) Prove that $\det(T - \lambda l_v) = \det([T]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

7. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   (b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

   (c) State and prove results analogous to (a) and (b) for matrices.

8. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

9. Let V be a finite-dimensional vector space, and let λ be any scalar.

   (a) For any ordered basis β for V, prove that $[\lambda l_V]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of λI_V.

(c) Show that λI_V is diagonalizable and has only one eigenvalue.

10. A **scalar matrix** is a square matrix of the form λI for some scalar λ;that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

    (a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

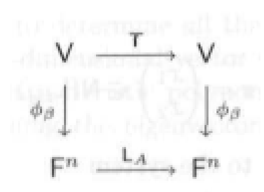
    (b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

    (c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

11. (a) Prove that similar matrices have the same characteristic polynomial.

    (b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

12. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [T]_\beta$. In reference to the following figure, prove the following.



    (a) If $v \in V$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.

    (b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in F^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

13. For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

14. (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

    (b) State and prove the analogous result for matrices.

15. (a) Prove that similar matrices have the same trace.
    Hint : $tr(AB) = tr(BA)$ and $tr(A) = tr(A^t)$.

    (b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

16. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.

    (a) Show that ± 1 are the only eigenvalues of T.

    (b) Describe the eigenvectors corresponding to each eigenvalue of T.

    (c) Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

    (d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.

17. Let $A, B \in M_{n\times n}(\mathbb{C})$.

    (a) Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible.
    Hint : Examine $\det(A + cB)$.

    (b) Find nonzero 2×2 matrices A and B such that both A and $A + cB$ are invertible for all $c \in \mathbb{C}$.

18. Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered bases β and γ for V such that $A = [T]_\beta$ and $B = [T]_\gamma$.

19. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0.$$

    (a) Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

    (b) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\ldots(A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n - 2$.
    Hint : Apply mathematical induction to n.

    (c) Show that $tr(A) = (-1)^{n-1}a_{n-1}$.

20. (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

    (b) State and prove a comparable result for matrices.

    (c) Verify (b) for the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbb{R}$ with polynomial $g(t) = 2t^2 - t + 1$, eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and corresponding eigenvalue $\lambda = 4$.

21. Use the above Exercise to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(T) = T_0$, the zero operator.

22. Determine the number of distinct characteristic polynomials of matrices in $M_{2\times 2}(\mathbb{Z}_2)$.

23. Do matrix-equivalent matrices have the same eigenvalues?

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